DISSERTATIO MATHEMATICA

DE

EVOLUTIS *SECTIONUM CONICARUM.*

QUAM

CONS. AMPL. FACULT. PHILOS. ABOËNS.

PRÆSIDE

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PRO LAUREA

Publice ventilandam sistit

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À

MONSIEUR

CHARLES ERIC SJÖMAN

ENSEIGNE À LA MARINE.

Agréez, je vous en supplie, Monsieur, ce faible hommage de ma reconnaissance, et des sentimens respectueu~~ses~~ avec lesquels j'ai l'honneur d'etre

MONSIEUR

VOTRE

très humble serviteur
JEAN GUSTAVE FLORIN.

§. 1.

Naturam Linearum Curvarum considerantes, facile invenimus, eas ejus esse indolis, ut, ductis ad diversa ipsarum puncta tangentibus, plus vel minus ab his ipsis tangentibus decedant arcus, nullamque præter circulum dari curvam, quæ eundem cum tangente semper efficiat angulum. Hinc quoque Circulum ceu mensuram Curvaturæ reliquarum Linearum Curvarum statuerunt Mathematici. Per quodvis videlicet punctum Curvæ cujusdam concipiatur Circulus, ipsam Curvam in puncto isto tangens, descriptus, qui itaque in hoc puncto Curvaturam exhibet. Quumque inæqualiter a tangente decedant Curvæ, Radii quoque Circuli tangentis vel osculatorii, ut etiam nuncupatur, inæquales evadunt, nec in idem punctum extremitates ipsorum coire possunt.

Hæc ipsa Radiorum Curvaturæ proprietas ansam suppeditavit Mathematicis, *Locum Geometricum* pro centris horum Radiorum investigandi, & HUGENIUS (*) primus naturam investigavit hujus Loci Geometrici, nomenque illi dedit *Evolutæ*, eo ex fundamento, quod, posita Evoluta *GOR* data, si filum *AGR* perfecte flexile illi circumplicetur, ita ut portione *AG* superet Longitudinem arcus Evolutæ datæ *GOR* & iterum successive ab ea abducatur, extremitas ejus *A* (extenso filo in rectam *MR*) curvam aliam *AM* describit; alteram itaque harum *AM* ex evolutione descriptam, alteramque *AGOR* Evolutam vocavit. Post illum, doctrinam de Evolutis omnes fere Geometriæ Sublimioris Cultores tractaverunt, quo factum est, ut formula jam exstet generalis, cujus ope, data æquatione Curvæ, dabitur æquatio Curvæ Evolutæ. Has vero Evolutarum æquationes in quovis casu a formula generali non absque omni difficultate deducere possumus; quamvis enim pro Parabola absque prolixo calculo determinari possit æquatio Evolutæ, res tamen æque facile pro Ellipsi atque Hyperbola non succedit. Quo itaque commodior ad æquationes Evolutarum Sectionum Conicarum inveniendas pateat via, in sequentibus specialem pro his Curvis solutionem, Speciminis Academici loco, adferre nobis proposuimus.

§. 2.

(*) Cfr. Histoire des Mathematiques par MONTUCLA. Tom. II. part. IV. Liv. II. p. 129.

§. 2.

LEMMA 1. Si ex quolibet puncto M cujusvis Sectionis Conicæ ducta sit Linea Normalis MN, quæ axi conveniat in N, & ex aliquo foco F, ducto ramo FM, in ipsum ex N ducatur perpendicularis ND, erit portio MD æqualis Semiparametro axis. Cfr. GVIDONIS GRANDI *Synopſ. Sect. Conicar. Propoſ.* 31.

LEMMA 2. In quavis Sectione Conica, si fuerit Parameter axis $= 2p$, $\wedge FMN = w$ & $Sin.\ tot. = 1$, erit $MN = \frac{p}{Cos\, w}$. In $\triangle\ MDN$ rectangulo est $MN : MD :: 1 : Cos\, DMN$ unde $MN = \frac{MD}{Cos\, DMN} = \frac{p}{Cos\, w}$ (Lemm. 1).

§. 3.

PROBLEMA. Si fuerit Curva AM sectio quædam Conica, cujus vertex A, Focus F & Axis AP, invenire æquationem Evolutæ pro hac Curva.

Sumto puncto quodam M, ducatur Radius Curvaturæ RM, eritque (§. 1) punctum R in Evoluta. Sit Parameter axis $= 2p$, erit EF, Perpendiculariter ducta a puncto F in AP, æqualis semi parametro $= p$ (Elem. Sect. Conic.) Facta $AG = p$ = Radio Curvaturæ in vertice Sectionis Conicæ, statuatur punctum G origo Abscissarum & $GP = x$, ductaque PR normaliter in AP sit $PR = y$; sumta præterea Pp infinite parva, ducatur pr parallela ipsi PR & RS parallela axi abscissarum AP, erit $Pp = RS = dx$ & $Sr = dy$. Posita $AF = m$, habebitur, existente Centro Ellipseos atque Hyperbolæ in C, axis harum Curvarum Major $AC = a = \frac{\pm m^2}{2m - p}$ & $FC = e = \frac{\pm m(p - m)}{2m - p}$. Sit deinde $> FMN = w$ & $PNR = \phi$, erit $RM = \frac{MN^3}{p^2}$ (Elem. Sect. Con.) $= \frac{p}{Cos\ w^3}$ (Lem. 2) & $Sin\ w = \frac{e}{a} Sin\ \phi$ = Sin φ $\frac{\pm p \mp m}{m}$. Ducta enim Ordinata MQ Curvæ AM, erit (GVID. GRANDI *Synops. Sect. Conic. pr. 33. Cor. 2*) $a^2 : e^2 :: CQ : CN$, unde $CN = \frac{e^2 CQ}{a^2}$ & $FN = \pm CF \mp CN = \pm e \frac{\mp e^2 CQ}{a^2}$. Est vero $FM = \pm a \frac{\mp e CQ}{a}$; si itaque ponatur $MF : FN :: a : e$ seu $\pm a \frac{\mp e CQ}{a} : \pm e \frac{\mp e^2 CQ}{a^2} :: a : e$, erit $\pm ae \frac{\mp e^2 CQ}{a} = \pm ae \frac{\pm e^2 CQ}{a}$, sumto videlicet producto extremorum ac mediorum terminorum, qua itaque æquatione veritas proportionis abunde constat. Erat vero $RM = \frac{MN^3}{p^2} = \frac{p}{Cos\ w^3}$, adeoque $RN = RM - MN = MN\left(\frac{MN^2}{p^2} - 1\right) = \frac{p}{Cos\ w^3}$

$-\frac{p}{Cos\,w} = p\left(\frac{1 - Cos\,w^2}{Cos\,w^3}\right) = \frac{p\,Sin\,w^2}{Cos\,w^3}$ Est autem in $\triangle$ *RPN*, *NR*: *PR*:: 1: *Sin* φ, unde $PR = y = NR\,Sin\varphi = \frac{p\,Sin\,w^2}{Cos\,w^3}\,Sin\,\varphi = \frac{p\,m\,Sin\,w^3}{(\pm p \mp m)\,Cos\,w^3} = \frac{pm\,Tg\,w^3}{\pm p \mp m}$, ex qua æqv. habebitur $Tg\,w = \frac{\sqrt[3]{(\pm p \mp m)y}}{p\,m}$ & posita $\frac{\pm p \mp m}{p\,m} = q$, erit $Tg\,w = q^{\frac{1}{3}}y^{\frac{1}{3}}$ & $Sin\,w = \left(\frac{Tg\,w}{\sqrt{1 + Tg\,w^2}}\right)$ $= \frac{q^{\frac{1}{3}}\;y^{\frac{1}{3}}}{\sqrt{1 + q^{\frac{2}{3}}\;y^{\frac{2}{3}}}}$. Ex supra demonstratis habetur

$Sin\,\varphi = \frac{m\,Sin\,w}{\pm p \mp m} = \frac{Sin\,w}{pq} = \frac{y^{\frac{1}{3}}}{p\,q^{\frac{2}{3}}\sqrt{1 + q^{\frac{2}{3}}\;y^{\frac{2}{3}}}}$; unde $Cos\,\varphi^2 = 1 - Sin\,\varphi^2 = 1 - \frac{y^{\frac{2}{3}}}{p^2\,q^{\frac{4}{3}}\,(1 + q^{\frac{2}{3}}\;y^{\frac{2}{3}})}$

$= \frac{p^2\,q^{\frac{4}{3}}\,(1 + q^{\frac{2}{3}}\;y^{\frac{2}{3}}) - y^{\frac{2}{3}}}{p^2\,q^{\frac{4}{3}}\,(1 + q^{\frac{2}{3}}\;y^{\frac{2}{3}})} = \frac{p^2\,q^{\frac{4}{3}} + (p^2\,q^2 - 1)\,y^{\frac{2}{3}}}{p^2\,q^{\frac{4}{3}}\,(1 + q^{\frac{2}{3}}\;y^{\frac{2}{3}})}$

& $Cos\,\varphi = \frac{\sqrt{p^2\,q^{\frac{4}{3}} + (p^2\,q^2 - 1)\,y^{\frac{2}{3}}}}{p\,q^{\frac{2}{3}}\sqrt{1 + q^{\frac{2}{3}}\;y^{\frac{2}{3}}}}$, Adeoque $Tg\,\varphi$

$= \frac{Sin\,\varphi}{Cos\,\varphi} = \frac{y^{\frac{1}{3}}}{p\,q^{\frac{2}{3}}\sqrt{1 + q^{\frac{2}{3}}\;y^{\frac{2}{3}}}} \times \frac{p\,q^{\frac{2}{3}}\sqrt{1 + q^{\frac{2}{3}}\;y^{\frac{2}{3}}}}{\sqrt{p^2\,q^{\frac{4}{3}} + (p^2\,q^2 - 1)\,y^{\frac{2}{3}}}}$

=

$= \frac{y^{\frac{1}{3}}}{\sqrt{p^2 q^{\frac{4}{3}} + (p^2 q^2 - 1) y^{\frac{2}{3}}}}$. In $\triangle$ *RSr* ad *S* rectangulo est $RS : Sr : 1 : Tg\phi$, seu $dx : dy :: 1 : \frac{y^{\frac{1}{3}}}{(p^2 q^{\frac{4}{3}} + (p^2 q^2 - 1) y^{\frac{2}{3}})^{\frac{1}{2}}}$ unde $dx = \frac{dy}{y^{\frac{1}{3}}} \sqrt{p^2 q^{\frac{4}{3}} + (p^2 q^2 - 1) y^{\frac{2}{3}}}$, ex qua æquatione integrando eruitur ratio inter coordinatas Evolutæ Orthogonales *x* & *y*. Seorsim vero in sequentibus pro quavis Sectione Conica in hujus æquationis integrale inquirere juvat.

SCHOL. Problematis inversi, quo ex data Evoluta *GOR* atque Longitudine fili *AGOR*, curva *AM* ex evolutione genita investigatur, solutio haud est difficilis.

Ducta etenim Linea *RH* ipsi *AP* parallela, producatur *MQ* ad *H* & ponatur $AQ = z$, $AG = b$, $MQ = v$, p
GOR $= s$, manentibus reliquis denominationibus; erit *PN* Subtangens Evolutæ $= \frac{ydx}{dy}$ atque *NR* ejusdem linea tangens ~~normalis~~, cujus valor generalis $= \frac{y(dx^2 + dy^2)^{\frac{1}{2}}}{dy}$, quarum ope determinantur $MH = v + y = \frac{MR. PR}{NR}$

=

$$= \frac{\overline{p+s}\, dy}{(dx^2+dy^2)^{\frac{1}{2}}} \;\&\; RH = PQ = \frac{RM.NP}{RN} = \frac{\overline{p+s}.\, dx}{(dx^2+dy^2)^{\frac{1}{2}}}.$$

Est autem $AQ = z = AG + GP - PQ = p + x - \frac{\overline{p+s}.\, dx}{(dx^2+dy^2)^{\frac{1}{2}}}$; adeoque ex data æquatione Evolutæ, secundum regulas consuetas eliminari possunt x & y, novaque exsurgit æquatio, relationem inter coordinatas z & v Curvæ AM orthagonales exhibens. *Cfr.* ABR. GOTTH. KÄSTNER *anfangsgründe der Analysis des unendlichen. Götting.* 1761. *p.* 529. 530.

§. 4.

Si fuerit Curva AM Parabola Conica, erit, ducta ordinata EF per Focum F, $EF^2 = p^2 = 2pm$ seu $p = 2m$, adeoque $p^2q^2 - 1 = \frac{p\,(p-2m)}{m} = 0$. Æquatio itaque differentialis $dx = \frac{dy}{y^{\frac{1}{3}}}\sqrt{p^2q^{\frac{4}{3}} + (p^2q^2-1)y^{\frac{2}{3}}}$ pro Evoluta Parabolæ in hanc redigitur formam: $dx = \frac{pq^{\frac{2}{3}}\, dy}{y^{\frac{1}{3}}}$, sumtisque integralibus habebitur, facta correctione, $x = 0 = y$, $x = \frac{3\,p\,q^{\frac{2}{3}}\,y^{\frac{2}{3}}}{2}$ seu $x^3 = \frac{27\,p^3\,q^2\,y^2}{8} = \frac{27\,p\,y^2}{8}$ ob $q^2 = \frac{1}{p^2}$ seu y^2

$= \frac{8 p^{-1} x^3}{27}$, quæ quidem est æquatio ad Parabolam Semi Cubicam, vel ut etiam nuncupatur, *Neillianam*. Uberiorem vero in naturam hujus Curvæ investigationem eo ex fundamento omittimus, quod satis sit cognita; dixisse sufficiet eam ejus esse indolis, ut ad distantiam a vertice A ipsius Parabolæ $=p$, seu in ipsa Origine abscissarum G, punctum habeat cuspidis.

SCHOL. Si jam ope Scholii in §. præc. allati investigetur curva ex evolutione genita, data parabolæ Semi cubicæ æquatione $ay^2 = x^3$, sequenti modo procedendum. Eruatur valor ipsius $y = \frac{x^{\frac{3}{2}}}{a^{\frac{1}{2}}}$ adeoque dy

$X^{\frac{1}{2}}$

$= \frac{3x^{\frac{1}{2}}\,dx}{2a^{\frac{1}{2}}}$ & $\frac{ydx}{dy} = \frac{x^{\frac{3}{2}}}{a^{\frac{1}{2}}}.dx.\frac{2a^{\frac{1}{2}}}{3xdx} = \frac{2x}{3}$ &

$y\left(\frac{dx^2 + dy^2}{dy}\right)^{\frac{1}{2}} = \frac{x^{\frac{3}{2}}}{a^{\frac{1}{2}}}\,dx.\left(1 + \frac{9x}{4a}\right)^{\frac{1}{2}}.\frac{2a^{\frac{1}{2}}}{3x^{\frac{1}{2}}dx} =$

$x\frac{(4a + 9x)^{\frac{1}{2}}}{(3a^{\frac{1}{2}})}$ adeoque $v = \frac{MR.\,PR}{NR} - y$ (§. 3. Schol.)

$= \frac{3x^{\frac{1}{2}}\,\overline{p + s}}{(9x+4a)^{\frac{1}{2}}} - y$ & $z = p + x - \frac{2a^{\frac{1}{2}}\,\overline{p + s}}{(9x + 4a)^{\frac{1}{2}}}$. Quumque

que semper sit $s = \int (dx^2 + dy^2)^{\frac{1}{2}} = \int dx \left(\frac{9x + 4a}{2a^{\frac{1}{2}}}\right)^{\frac{1}{2}}$

$= \left(\frac{9x + 4a}{27a^{\frac{1}{2}}}\right)^{\frac{3}{2}} - \frac{8}{27} a$ facta debita correctione, habebitur, substituendo loco s ipsius valor jam determinatus, $v = 3x^{\frac{1}{2}} \frac{(27a^{\frac{1}{2}} p + \overline{9x+4a}^{\frac{3}{2}} - 8a^{\frac{3}{2}})}{27a^{\frac{1}{2}} (9x4a)^{\frac{1}{2}}} - y$

(A) & $z = p + x - 2^{\frac{1}{2}} \frac{(27a^{\frac{1}{2}} p + \overline{9x+4a}^{\frac{3}{2}} - 8a)^{\frac{3}{2}}}{27^{\frac{1}{2}} (9x+4a)^{\frac{1}{2}}}$

(B) unde, comparando æquat. (A) cum æquat. Evolutæ $x^3 = ay^2$, (C) exterminatur y, & pariter comparando æquat. (B) cum æquat. (C) eliminatur x, quibus demum æquatio relationem inter v & z exhibens determinatur. Hinc vero jam videtur, varias pro diverso ipsius p valore exstare Curvas, omnes quidem inter se parallelas, diversæ tamen indolis. Si ponatur $27p = 8a$, æquat. (A) in hanc reducitur formam; $v = \frac{4a^{\frac{1}{2}} x^{\frac{1}{2}}}{9}$ & $z = p + x - \frac{2}{3}x - \frac{8a}{27} = \frac{1}{3}x$, unde exterminando x, habebitur $v^2 = \frac{16az}{27}$ seu æquatio ad Parabolam Apollonianam cujus parameter $= \frac{16}{27}a$. Cfr. KÄSTNER *l. c.* Eodem modo res se quoque habet cum reliquis Curvis, ex evolutione Evolutarum Sectionum Conicarum genitis, quapropter

 ube-

uberiorem hujus rei explicationem in sequentibus omittimus.

§. 5.

In casu quo arcus AM fuerit portio Ellipseos, vel Hyperbolæ, Evoluta harum Curvarum habebitur integrando æquationem: $dx = \frac{dy\sqrt{p^2 q^{\frac{4}{3}} + (p^2 q^2 - 1)y^{\frac{2}{3}}}}{y^{\frac{1}{3}}}$ eritque $x + \frac{p^3 q^2}{p^2 q^2 - 1} = \left(\frac{p^2 q^{\frac{4}{3}} + \overline{p^2 q^2 - 1}\, y^{\frac{2}{3}}}{p^2 q^2 - 1}\right)^{\frac{3}{2}}$ æquatione ita correcta ut simul sit $x = o = y$. Si itaque loco quantitatis q, substituatur valor ipsius, in Ellipsi $= \frac{p - m}{pm}$, habebitur, ductis singulis terminis in $\frac{p.\overline{p - 2m}}{m^2}$ æquatio Evolutæ $\frac{p.\overline{p - 2m}.x}{m^2} + \frac{\overline{p - m}^2}{m^2} = \left(\frac{p^{\frac{2}{3}}.\overline{p - m}^{\frac{4}{3}}}{m^{\frac{4}{3}}} + \frac{p(p - 2m)\, y^{\frac{2}{3}}}{m^2}\right)^{\frac{3}{2}}$. Sumendo vero quadratum hujus æquationis, prodit, evolutis terminis, $\frac{p^2.\overline{p - 2m}^2.x^2}{m^4} + \frac{2p^2.\overline{p - 2m}.\overline{p - m}.x}{m^4} + \frac{p^2\overline{p - m}^4}{m^4} = \frac{p^2.\overline{p - m}^4}{m^4} + \frac{3p^{\frac{7}{3}}.\overline{p - m}^{\frac{8}{3}}.\overline{p - 2m}.y^{\frac{2}{3}}}{m^{\frac{14}{3}}}$ +

$$+\frac{3p^{\frac{8}{3}}.\overline{p-m}^{\frac{4}{3}}.\overline{p-2m}^{2}.y^{\frac{4}{3}}}{m^{\frac{16}{3}}}+\frac{p^{3}.\overline{p-2m}^{3}.y^{2}}{m^{6}},$$

illamque per m^4 multiplicando & $p^2.\overline{p-2m}$ dividendo eruitur $\overline{p-2m.x}^{2}+\overline{2p-m.x}-\frac{p.\overline{p-2m}^{2}y^{2}}{m^{2}}$

$$=\left(\frac{3p^{\frac{1}{3}}.\overline{p-m}^{\frac{4}{3}}y^{\frac{2}{3}}}{m^{\frac{2}{3}}}\right)\left(\frac{\overline{p-m}^{\frac{4}{3}}+p^{\frac{1}{3}}.\overline{p-2m}.y^{\frac{2}{3}}}{m^{\frac{2}{3}}}\right).$$

Hinc autem jam videre licet, curvam hanc in ſpatio finito conſtitutam eſſe, & quatuor habere puncta Cuſpidum, quorum unum in axi Ellipſeos majori ad diſtantiam a vertice = p, alterum vero in puncto quo Evoluta axem minorem tangit, & reliqua in regione abſciſſarum negativa, ad eandem a centro diſtantiam, qua ſita ſunt puncta jam nominata. Hoc vero calculo ita evinci poteſt: ex æquatione

$$\frac{p.\overline{p-2m}}{m^{2}}.x+\frac{p.\overline{p-m}^{2}}{m^{2}}=\left(\frac{p^{\frac{2}{3}}.\overline{p-m}^{\frac{4}{3}}}{m^{\frac{4}{3}}}+\frac{p.\overline{p-2m}.y^{\frac{2}{3}}}{m^{2}}\right)^{\frac{3}{2}},$$

poſitis brevitatis cauſſa $\frac{p.\overline{p-2m}}{m^{2}}=B$ & $\frac{p^{\frac{2}{3}}.\overline{p-m}^{\frac{4}{3}}}{m^{\frac{4}{3}}}=A$,

ſumatur valor ipſius $y=((Bx+A^{\frac{3}{2}})^{\frac{2}{3}}-A)^{\frac{3}{2}}$ &

$$dy=\frac{((Bx+A^{\frac{3}{2}})^{\frac{2}{3}}-A)^{\frac{1}{2}}Bdx}{(Bx+A^{\frac{3}{2}})^{\frac{1}{3}}}\text{ atque}$$

$ddy = \frac{AB^2\,dx^2}{3(Bx+A^{\frac{3}{2}})^{\frac{4}{3}}\,((Bx+A^{\frac{3}{2}})^{\frac{2}{3}}-A)^{\frac{1}{2}}}$. Hæc expressio secundum regulas in Geometria sublimiori traditas, nihilo æqualis est statuenda, quo facto, duo eruuntur factores $Bx+A^{\frac{3}{2}}=o$ & $(Bx+A^{\frac{3}{2}})^{\frac{2}{3}}-A=o$ unde $x=-\frac{A^{\frac{3}{2}}}{B}$ & $x=o$, quibus intelligitur alterum punctum cuspidis in ipsa origine abscissarum G, alterum vero ad distantiam ab hac origine $=-\frac{A^{\frac{3}{2}}}{B}$, quam, restitutis ipsarum A & B valoribus, æqualem invenimus $-\left(\frac{\overline{p-m}^2}{p-2m}\right)=\frac{\overline{p-m}^2}{2m-p}=\frac{m^2}{2m-p}-p=a-p=GC$. Sed ducta ordinata per punctum C, incidit hæc ipsa in axem Ellipseos minorem, adeoque alterum punctum Cuspidis erit in eo puncto, quo Evoluta axem minorem tangit.

Hanc Curvam absolute rectificabilem esse, exinde patet, quod semper sit æqualis differentiæ inter Radium Curvaturæ & semiparametrum axis majoris $=R-p$.

§. 6.

Quo vero pateat natura Evolutæ Hyperbolæ, læ, resumatur Æqu. §. 5. inventa $x + \frac{p^3q^2}{p^2q^2-1} = \left(\frac{p^2q^{\frac{4}{3}} + (p^2q^2-1)y^{\frac{2}{3}}}{p^2q^2-1}\right)^{\frac{3}{2}}$, quæ ducta in p^2q^2-1, dabit $(p^2q^2-1)x + p^3q^2 = (p^2q^{\frac{4}{3}} + (p^2q^2-1)y^{\frac{2}{3}})^{\frac{3}{2}}$ cujus deinde quadratum sumendo, prodit, terminis rite evolutis $(p^2q^2-1)^2x^2 + 2(p^2q^2-1)p^3q^2x + p^6q^4 = p^6q^4 + (3p^2q^{\frac{4}{3}}(p^2q^2-1)y^{\frac{2}{3}})(p^2q^{\frac{4}{3}} + (p^2q^2-1)y^{\frac{2}{3}}) + (p^2q^2-1)^3y^2$ & divisa æquatione per p^2q^2-1, factaque debita reductione habebitur $(p^2q^2-1)x^2 + 2p^3q^2x = (3p^2q^{\frac{4}{3}}y^{\frac{2}{3}})(p^2q^{\frac{4}{3}} + (p^2q^2-1)y^{\frac{2}{3}}) + \overline{p^2q^2-1}^2.\,y^2$ Erat autem pro Hyperbola quantitas $q =$ (§. 3.) $\frac{m-p}{pm}$ adeoque $p^2q^2-1 = \frac{p(p-2m)}{m^2}$ unde facta substitutione erit $\frac{p(p-2m)}{m^2}x^2 + \frac{2p.\overline{m-p}^2.x}{m^2}$

$$= \frac{3\overline{m-p}^{\frac{4}{3}}p^{\frac{2}{3}}y^{\frac{2}{3}}}{m^{\frac{4}{3}}}\left(\frac{\overline{m-p}^{\frac{4}{3}}p^{\frac{2}{3}}}{m^{\frac{4}{3}}} + \frac{p.(p-2m)y^{\frac{2}{3}}}{m^2}\right)$$

$+ \frac{p^2(p-2m)^2y^2}{m^4}$ & ducta æquatione in $\frac{m^2}{p}$ prodit æquatio Evolutæ $\overline{p-2m}.\,x^2 + 2\,\overline{m-p}^2\,x$

$$= \frac{3\, m^{\frac{2}{3}}\, \overline{m - p^{\frac{4}{3}}}\, y^{\frac{2}{3}}}{p^{\frac{1}{3}}} \left(\frac{\overline{m - p^{\frac{4}{3}}}\, p^{\frac{2}{3}}}{m^{\frac{3}{4}}} + \frac{p\,(p-2m)\, y^{\frac{2}{3}}}{m^2} \right)$$
$$+ \frac{p\, \overline{p - 2m}^{\,2}\, y^2}{m^2}.$$

Determinata vero jam æquatione Curvæ, ad indolem ipsius investigandam pergimus; & videndum nobis primo erit, an & qualia puncta singularia hæc habeat curva. Existente itaque $y = ((Bx + A^{\frac{3}{2}})^{\frac{2}{3}} - A)^{\frac{3}{2}}$ factis brevitatis causa $\frac{p.\overline{p-2m}}{m^2} = B$ & $\frac{p^{\frac{2}{3}}.\, \overline{m - p^{\frac{4}{3}}}}{m^{\frac{4}{3}}} = A$, habebitur $dy = \frac{((Bx + A^{\frac{3}{2}})^{\frac{2}{3}} - A)^{\frac{1}{2}}}{(Bx + A^{\frac{3}{2}})^{\frac{1}{3}}} Bdx$ &

$$ddy = \frac{AB^2 dx^2}{3\,(Bx + A^{\frac{3}{2}})^{\frac{4}{3}}\,((Bx + A^{\frac{3}{2}})^{\frac{2}{3}} - A)^{\frac{1}{2}}} = o$$

unde duo erui possunt factores $(Bx + A^{\frac{3}{2}})^{\frac{4}{3}} = o$ & $((Bx + A^{\frac{2}{3}})^{\frac{3}{2}} - A^{\frac{3}{2}})^{\frac{1}{2}} = o$ quorum alter indicat punctum Cuspidis esse in ipsa origine abscissarum G, ubi $x = o$ & alter ejusdem generis punctum inveniri in Hyperbola opposita, ad distantiam a centro $= -\frac{A^{\frac{3}{2}}}{B}$

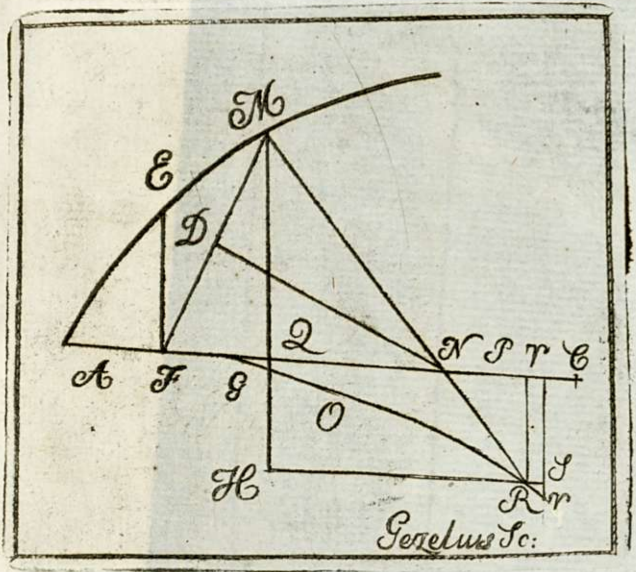
$$= -\frac{(m-p)^2}{p-2m} = \frac{(m-p)^2}{2m-p} = \frac{m^2}{2m-p} - p = -a - p.$$

Ex

Ex allatis liquet, Curvam hanc quatuor ramos in infinitum excurrentes, habere; an vero etiam Asymptotos haheant rami, calculo nobis jam est investigandum. Calculus Sublimior tales nobis exhibet formulas pro invenda origine Asymptotorum, ut sumto valore $\frac{\pm ydx}{dy} \mp x$ (prout ipsa Curva respectu Axis Abscissar. Concava fuerit vel convexa) in terminis ipsius x, pariter ac ipsius $y - \frac{xdy}{dy}$, statuatur $x = \infty$, si valor finitus supersit, hic idem valor determinabit punctum, e quo prodeunt Asymptoti, & angulum, quem cum Axe Abscissarum efficiunt. Ex æquatione autem Evolutæ habebitur, retentis iisdem denominationibus ac antea $y = ((Bx + A^{\frac{3}{2}})^{\frac{2}{3}} - A)^{\frac{3}{2}}$, adeoque $dy = \frac{B((Bx + A^{\frac{3}{2}})^{\frac{2}{3}} - A)^{\frac{1}{2}} dx}{(Bx + A^{\frac{3}{2}})^{\frac{1}{3}}}$ unde $x - \frac{ydx}{dy}$

$$= x - ((Bx + A^{\frac{3}{2}})^{\frac{2}{3}} - A)^{\frac{3}{2}} dx. \frac{(Bx + A^{\frac{3}{2}})^{\frac{1}{3}}}{B(Bx + A^{\frac{3}{2}})^{\frac{2}{3}} - A)^{\frac{1}{2}} dx}$$

$$= x - \frac{((Bx + A^{\frac{3}{2}})^{\frac{2}{3}} - A)(Bx + A^{\frac{3}{2}})^{\frac{1}{3}}}{B}$$; quæ expressio, posita $x = \infty$, evadit in hanc formam: $x - \frac{B^{\frac{2}{3}} x^{\frac{2}{3}}}{B} B^{\frac{1}{3}} x^{\frac{1}{3}} = Bx - Bx$, quæ, cum infinita sit, originem Asymptotorum exhibere non potest. Neque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que situm ipsarum ope alterius jam allatæ formulæ $y - \frac{xdy}{dx}$ determinare possumus. Facta etenim substitutione, erit $y - \frac{xdy}{dx} = y - \frac{Bx((Bx+A^{\frac{3}{2}})^{\frac{2}{3}} - A)^{\frac{1}{2}} dx}{(Bx+A^{\frac{3}{2}})^{\frac{1}{3}} dx}$

$$= ((Bx+A^{\frac{3}{2}})^{\frac{2}{3}} - A)^{\frac{3}{2}} - \frac{Bx((Bx+A^{\frac{3}{2}})^{\frac{2}{3}} - A)^{\frac{1}{2}}}{(Bx+A^{\frac{3}{2}})^{\frac{1}{3}}},$$

unde, statuendo $x = \infty$, eruitur $Bx - \frac{Bx . B^{\frac{1}{3}} x^{\frac{1}{3}}}{B^{\frac{1}{3}} x^{\frac{1}{3}}} = Bx - Bx$, quibus intelligitur, Curvam hanc Asymptotis destitutam esse.

Quod vero ad rectificationem hujus Curvæ attinet, illam facillime esse inveniendam, ex inde patet, quod semper sit ipsa Curva $= R - p$.